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A complete solution of a constrained system: SUSY monopole quantum mechanics

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Abstract: We solve the quantum mechanical problem of a charged particle on S^2 in the background of a magnetic monopole for both bosonic and supersymmetric particles by constructing Hilbert space and realizing the fundamental operators obeying complicated Dirac bracket relations in terms of differential operators. We find the complete energy eigenfunctions. Using the lowest energy eigenstates we count the number of degeneracies and examine the supersymmetry structure of the ground states in detail.

Keywords: Spontaneous Symmetry Breaking, Sigma Models, Supersymmetric gauge theory, Solitons Monopoles and Instantons.

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1. Introduction

The nonlinear sigma model and its supersymmetric generalization have provided widespread applications in lower dimensional field theory [1], string theory in curved spacetime [2] and supersymmetric quantum mechanics [3], and their quantization has been an important issue in theoretical physics. One of the methods to approach the model is to use the constrained variables. In such case, it is well known that the canonical method has to be replaced by the Dirac procedure [4]. However, it is very hard, in general, to construct the Hilbert space and explicitly realize the operators as differential operators due to highly nonlinear nature of the Dirac approach. Since there does not exist a general method to deal with such problems, each case has to be treated separately. One of the purposes of this paper is to demonstrate a complete quantization procedure for certain particular supersymmetric nonlinear sigma models.

The system we are interested in is a supersymmetric quantum mechanical particle moving on S^2 under the influence of Dirac magnetic monopole [5] located at the center. Since the work of Wu and Yang [6] revealing that a charged particle interacting with a magnetic monopole can be described in terms of the monopole harmonics and the system exhibits many interesting features [7], several supersymmetric models have been proposed and various physical aspects of the supersymmetric generalization have been investigated [8 – 11]. In particular, it was discovered [8] that the system in R^3 admits $N = 1$ supersymmetric generalization. It was then found that the model has another hidden supersymmetry and pointed out that this additional supersymmetry is related to the model restricted to S^2 [9]. $N = 2$ supersymmetric model on S^2 was studied in ref. [10] and the complete energy eigenfunctions were obtained. More recently, in refs. [12, 13], $N = 2$ and $N = 4$ supersymmetric models were studied in detail using the $CP(1)$ model type of variables, and certain issues related to the spontaneous supersymmetry breaking were discussed. This choice of variables was adopted because it allows relatively simple supersymmetric formulations of

the model and consequently $N = 2$ and $N = 4$ models could be treated in similar fashion. It also has the usual merits that the vector potential for the magnetic monopole and consequently the Lagrangian is free of singularity [14] and one does not have to deal with multi-valued action [15]. However, in these references the Hilbert space representation of the commutator algebra of the basic quantum observables was lacking.

In this paper, we fill this gap by constructing the Hilbert space by means of single valued functions on S^3 instead of on S^2 and finding the differential operator representation of the quantum observables. In the bosonic and $N = 2$ cases, the solution to the problem is well known. However, in $N = 4$ case the Hilbert space representation of the quantum operators and the complete energy eigenfunctions had not been given in the literature as far as we know. Furthermore, we show that the complete energy eigenfunctions can be expressed in terms of certain simple polynomials of z_i and $\bar{z}_i$, $i = (1, 2)$. This result, as a byproduct, provides another way of writing the monopole harmonics usually expressed in terms of the Jacobi polynomials. More importantly, using the exact energy eigenfunctions we count the number of ground state degeneracies and study the supersymmetric structure of the ground state energy sector, and discuss the important issue of spontaneous supersymmetry breaking in great detail. We also show that the Hamiltonian and the angular momentum operators are related in such a way that generalizes the classical results [7]: The minimum angular momentum quantum number $k_{\min} = |g|$ of a unit charged bosonic particle in the background of the monopole of strength g is replaced by $k_{\min} = |g - \sigma|$ in the case of a supersymmetric particle, where σ is the total spin component of the particle along the radial direction.

This paper is organized as follows. In section 2, we analyze the bosonic case using the $CP(1)$ model approach [16]. In section 3, the analysis is extended to $N = 2$ supersymmetric case, and in section 4 to the $N = 4$ case. The summary and discussions are given in section 5.

2. Bosonic particle

We start with the bosonic case. Although the quantization in this case is well known in terms of the usual variables [6, 10], we here present a detailed quantization procedure because the complete quantization using $CP(1)$ type of variables are not well known. Furthermore, the following sections on the supersymmetric particles will rest heavily on the results of this section. Also, the operator ordering ambiguity arising in the quantization procedure is carefully treated.

For notational convenience, we set the electric charge $q = -1$ and the mass of the particle $m = 1$. Although the particle is moving on S^2 (of unit radius), we will work with S^3 (also of unit radius), which is a principal $U(1)$ bundle over S^2. To be concrete, let us describe S^3 by two complex functions (z_1, z_2) satisfying $\bar{z} \cdot z \equiv \sum_{i=1}^{2} |z_i|^2 = 1$. The $U(1)$ group action on S^3 is given by $z \to \mathrm{e}^{i\Lambda} z$, and the base manifold is S^2. The projection map is given by $x^a \equiv \bar{z}\sigma^a z$ $(a = 1, 2, 3)$, which satisfy $x^a x^a = 1$, and σ^a denote Pauli matrices. From the fiber bundle point of view, dynamics of a particle moving on S^2 is described by the action of the form $A[z(t)]$ which is invariant under the local $U(1)$ transformation. This

way of writing the action in terms of S^3 coordinates instead of S^2 coordinates has certain mathematical advantages. For instance, the vector potential for the magnetic monopole has no string singularity when regarded as a field on S^3.

We write the Lagrangian as

$$L = 2|\dot{z} - z(\bar{z}\cdot\dot{z})|^2 + ig(\bar{z}\cdot\dot{z} - \dot{\bar{z}}\cdot z). \tag{2.1}$$

The first term is the kinetic part. It is invariant under the local $U(1)$ transformation and reduces to the standard kinetic energy term when written in terms of the S^2 coordinates. The second term represents the interaction of the particle with the magnetic monopole of strength g located at the center of S^2. Under the local $U(1)$ transformation, it changes only by a total time derivative term and the corresponding action is $U(1)$-invariant. In every respect, $U(1)$ plays the role of the electromagnetic gauge group. The interaction term again reduces to the familiar form (up to a gauge transformation) when expressed in terms of S^2 coordinates.

Let p and $\bar{p}$ denote the momenta conjugate to the fields z and $\bar{z}$ respectively,

$$p = 2\left(\dot{\bar{z}} - (\dot{\bar{z}}\cdot z)\bar{z}\right) + ig\bar{z}\,, \qquad \bar{p} = 2\left(\dot{z} - z(\bar{z}\cdot\dot{z})\right) - igz. \tag{2.2}$$

Due to the constraint $\bar{z}\cdot z = 1$, the momenta should satisfy

$$p\cdot z = ig\,, \qquad \bar{z}\cdot\bar{p} = -ig. \tag{2.3}$$

Using the constraints, the Hamiltonian can be written as

$$H_c = 2|\dot{z} - (\bar{z}\cdot\dot{z})z|^2 = \frac{1}{2}p_i A_{ij}\bar{p}_j, \tag{2.4}$$

where A_{ij}, defined by

$$A_{ij} \equiv \delta_{ij} - z_i\bar{z}_j, \tag{2.5}$$

satisfies

$$A_{ij}z_j = 0\,, \qquad \bar{z}_i A_{ij} = 0\,, \qquad A_{ij}A_{jk} = A_{ik}\,, \qquad \bar{A}_{ij} = A_{ji}. \tag{2.6}$$

The standard Poisson brackets are

$$\{z_i, p_j\} = \{\bar{z}_i, \bar{p}_j\} = \delta_{ij}, \tag{2.7}$$

with the remaining brackets being zero. A simple analysis shows that the constraints can be classified into the following two second class constraints

$$C_1 = \bar{z}\cdot z - 1\,, \qquad C_2 = \bar{z}\cdot\bar{p} + p\cdot z, \tag{2.8}$$

and one first class constraint

$$C_0 = -i(\bar{z}\cdot\bar{p} - p\cdot z) + 2g, \tag{2.9}$$

generating the $U(1)$ transformation. Because of the second class constraints we need to calculate the Dirac brackets according to the formula

$$\{A, B\}_D = \{A, B\} - \{A, C_a\}\Theta^{ab}\{C_b, B\}, \tag{2.10}$$

where Θ^{ab} is the inverse matrix of $\Theta_{ab} = \{C_a, C_b\}$. The result can be summarized as

$$\begin{aligned} \{p_i, z_j\}_D &= -\delta_{ij} + \frac{1}{2}\bar{z}_i z_j, \qquad \{p_i, \bar{z}_j\}_D = \frac{1}{2}\bar{z}_i\bar{z}_j, \\ \{p_i, p_j\}_D &= \frac{1}{2}(p_i\bar{z}_j - p_j\bar{z}_i), \qquad \{\bar{p}_i, p_j\}_D = \frac{1}{2}(\bar{z}_j\bar{p}_i - z_i p_j)\,. \end{aligned} \tag{2.11}$$

The canonical quantization proceeds by replacing the classical variables by the corresponding quantum operators[1], imposing the commutation relations according to the Dirac quantization rule, $\{A, B\}_D \to -i[A, B]$ and the complex conjugation becoming the Hermitian adjoint. In this step, there usually appears the notorious problem of operator ordering ambiguity. In our case, however, the ordering is fixed as follows:

$$\begin{aligned} [p_i, z_j] &= -i\delta_{ij} + \frac{i}{2}\bar{z}_i z_j, \qquad [p_i, \bar{z}_j] = \frac{i}{2}\bar{z}_i\bar{z}_j, \\ [p_i, p_j] &= \frac{i}{2}(p_i\bar{z}_j - p_j\bar{z}_i), \qquad [\bar{p}_i, p_j] = \frac{i}{2}(\bar{z}_j\bar{p}_i - z_i p_j). \end{aligned} \tag{2.12}$$

The above brackets should be supplemented by their Hermitian adjoints and all trivial commutation relations were omitted. Note that the brackets in the first line have no operator ordering ambiguity. In the second line, the ordering of the first bracket is fixed by the anti-symmetry property, while the ordering in the second bracket is fixed by requiring that the variables $(z_i, \bar{z}_i, p_i, \bar{p}_i)$ commute with the second class constraint, $C_2 = 0$. This choice of ordering appeared before in the bosonic $CP(1)$ model [17]. Next, we need to quantize the constraints. Obviously, C_1 has no ordering ambiguity. It can also be shown that C_2 is free from ambiguity if we demand it be self-adjoint. First class constraint C_0, however, suffers from the ordering ambiguity. Therefore, the quantum Gauss law constraint should be of the form

$$\hat{C}_0 \equiv -i(\bar{z}\cdot\bar{p} - p\cdot z) + \alpha_G + 2g = 0, \tag{2.13}$$

where the real constant α_G denotes the ordering parameter.

Before trying to find the Hilbert space representation of Eq. (2.12) it is useful to decompose p_i into two parts by introducing U_B and B_i as follows:

$$U_B \equiv -i(\bar{z}\cdot\bar{p} - p\cdot z)\,, \qquad B_i \equiv p_i + \frac{i}{2}U_B\bar{z}_i. \tag{2.14}$$

$\bar{B}_i$ is defined to be the Hermitian conjugate of B_i. The second class constraints (2.8) become

$$\bar{z}\cdot z - 1 = 0\,, \qquad B\cdot z = 0\,, \qquad \bar{z}\cdot\bar{B} = 0, \tag{2.15}$$

and the quantum Gauss law constraint (2.13) can be written as

$$U_B + \alpha_G + 2g = 0. \tag{2.16}$$

[1]In this paper we denote the quantum operators and the Hermitian adjoint by the same symbols as the corresponding classical quantities and the complex conjugate. The distinction should be clear from the context.

The basic commutation relations (2.12) can be rewritten, in terms of $(B_i, \bar{B}_i, U_B, z_i, \bar{z}_i)$, as follows:

$$\begin{aligned}
&[U_B, z_i] = z_i, && \left[U_B, \bar{B}_i\right] = \bar{B}_i, \\
&[B_i, z_j] = -iA_{ji}, && [B_i, \bar{z}_j] = 0, \\
&[B_i, B_j] = -i(\bar{z}_i B_j - \bar{z}_j B_i), && \left[B_i, \bar{B}_j\right] = -\left(U_B - \tfrac{1}{2}\right) A_{ji},
\end{aligned} \tag{2.17}$$

where A_{ji} was defined in Eq. (2.5). Again, we omitted the Hermitian adjoint and trivial relations. Next, we need to define the quantum Hamiltonian. There is again an ordering ambiguity. However, since a different choice of ordering in our model produces only a constant term upon imposing the Gauss law constraint, it suffices to choose one. We choose the following Hamiltonian:

$$\begin{aligned}
H &= \frac{1}{4}\left(p_i A_{ij} \bar{p}_j + \bar{p}_j A_{ij} p_i\right) \\
&= \frac{1}{4}\left(B_k \bar{B}_k + \bar{B}_k B_k - 1\right).
\end{aligned} \tag{2.18}$$

The angular momentum operator K_a is defined by

$$K_a = \frac{i}{2}\left(\bar{z}\sigma_a \bar{B} - B\sigma_a z + 2i\,\bar{z}\sigma_a z\right) - \frac{1}{2}\left(U_B - \frac{3}{2}\right)\bar{z}\sigma_a z, \tag{2.19}$$

which satisfies the following commutation relations:

$$\begin{aligned}
&[K_a, z_i] = -\frac{1}{2}(\sigma_a z)_i, && [K_a, \bar{z}_i] = \frac{1}{2}(\bar{z}\sigma_a)_i, \\
&\left[K_a, \bar{B}_i\right] = -\frac{1}{2}(\sigma_a \bar{B})_i, && [K_a, B_i] = \frac{1}{2}(B\sigma_a)_i, \\
&[K_a, U_B] = 0, && [K_a, K_b] = i\epsilon_{abc} K_c.
\end{aligned} \tag{2.20}$$

Its square turns out to be related to the Hamiltonian as

$$K_a K_a = 2H + \frac{1}{4}\left(U_B - \frac{3}{2}\right)^2. \tag{2.21}$$

To construct the Hilbert space, we first consider the functions of the form:

$$f(z, \bar{z}) = c_{i_1 \cdots i_m j_1 \cdots j_n} z_{i_1} \cdots z_{i_m} \bar{z}_{j_1} \cdots \bar{z}_{j_n}, \tag{2.22}$$

where (m, n) are non-negative integers. The complex coefficients $c_{i_1 \cdots i_n j_1 \cdots j_m}$ are totally symmetric with respect to the interchange of any two indices belonging to the same index group. Furthermore, we choose them to vanish when indices from different groups are contracted.[2] Such functions with a fixed pair of integers (m, n) generate a complex vector

[2]This is because if such two indices have a non-trivial trace the function can be reduced using the constraint $\bar{z} \cdot z - 1 = 0$. Thus, the above restriction on the coefficients can be regarded as a kind of irreducibility condition.

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bundle over S^2 of (m,n) type. The Hilbert space is defined as the direct sum of all such complex vector bundles. Hermitian inner product is given by

$$\langle f_1, f_2 \rangle = \int f_1^*(z,\bar{z}) f_2(z,\bar{z}) d\mu, \tag{2.23}$$

where

$$d\mu = \frac{1}{2\pi}\delta(\bar{z}\cdot z - 1) d\bar{z}_1 dz_1 d\bar{z}_2 dz_2. \tag{2.24}$$

With a straightforward calculation it can be shown that this integral is the usual integral on the base manifold times the integral over the $U(1)$ fiber.

On this Hilbert space we represent z_i and $\bar{z}_i$ as multiplications and B_i, $\bar{B}_i$ as follows:

$$\begin{aligned} B_i &= -iA_{kj}\frac{\partial}{\partial z_k}A_{ji} = -iA_{ki}\frac{\partial}{\partial z_k} + i\bar{z}_i, \\ \bar{B}_i &= -iA_{ij}\frac{\partial}{\partial \bar{z}_k}A_{jk} = -iA_{ik}\frac{\partial}{\partial \bar{z}_k}. \end{aligned} \tag{2.25}$$

It can be shown that $\bar{B}_i$ is the Hermitian adjoint of B_i with respect to the product (2.23), and that they satisfy the constraint (2.15). A further calculation shows that they reproduce the commutator algebra (2.17) if we represent U_B by

$$U_B = z_k\frac{\partial}{\partial z_k} - \bar{z}_k\frac{\partial}{\partial \bar{z}_k} + \frac{3}{2}. \tag{2.26}$$

Using this result, representation for other composite quantities can be easily found. For instance, the angular momentum can be represented as

$$K_a = \frac{1}{2}\left(\bar{z}\sigma_a\frac{\partial}{\partial \bar{z}} - \frac{\partial}{\partial z}\sigma_a z\right). \tag{2.27}$$

The physical states are those satisfying the Gauss law constraint (2.16), which we write as

$$\tilde{U}_B + 2\tilde{g} = 0, \tag{2.28}$$

where

$$\tilde{U}_B \equiv U_B - \frac{3}{2}, \qquad 2\tilde{g} \equiv \frac{3}{2} + \alpha_G + 2g. \tag{2.29}$$

Therefore, they are represented by the functions (2.22) with (m,n) satisfying

$$(m-n) + 2\tilde{g} = 0. \tag{2.30}$$

Note that $2\tilde{g}$ must be an integer. This implies that α_G should be a half integer if $2g$ is an integer according to the Dirac quantization condition of the monopole charge.

In order to obtain the energy spectrum we introduce the following operators:

$$a = \epsilon_{ij}B_i\bar{z}_j, \qquad \bar{a} = \epsilon_{ij}z_j\bar{B}_i, \tag{2.31}$$

in terms which the Hamiltonian (2.18) can be written as

$$H = \frac{1}{4}(a\bar{a} + \bar{a}a), \tag{2.32}$$

where

$$[a, \bar{a}] = -\tilde{U}_B\,, \qquad \left[\tilde{U}_B, a\right] = -2a\,, \qquad \left[\tilde{U}_B, \bar{a}\right] = 2\bar{a}. \tag{2.33}$$

Explicit differential operator representation of a and $\bar{a}$ is obtained by inserting eq. (2.25) into eq. (2.31):

$$a = -i\epsilon_{ij}\bar{z}_j\frac{\partial}{\partial z_i}\,, \qquad \bar{a} = -i\epsilon_{ij}z_j\frac{\partial}{\partial \bar{z}_i}, \tag{2.34}$$

and the Hamiltonian can be written as

$$H = \frac{1}{4}\left(-2\frac{\partial}{\partial z_i}\frac{\partial}{\partial \bar{z}_i} + 2z_i\frac{\partial}{\partial z_i}\bar{z}_j\frac{\partial}{\partial \bar{z}_j} + \bar{z}_i\frac{\partial}{\partial \bar{z}_i} + z_i\frac{\partial}{\partial z_i}\right). \tag{2.35}$$

When applied to the functions (2.22), the first term involving the second order derivatives vanishes due to the irreducibility property we required on the wavefunctions and the remaining terms give the following energy spectrum:

$$E = \frac{1}{4}(2mn + m + n), \tag{2.36}$$

with m and n related to each other by eq. (2.30). If $\tilde{g} \geq 0$, m can be any non-negative integer. So, we set $m = s$, $(s = 0, 1, 2, \cdots)$ and $n = s + 2\tilde{g}$. If $\tilde{g} \leq 0$, on the other hand, the roles of m and n are interchanged and we set $n = s$, $m = s - 2\tilde{g}$. Using this notation, the energy spectrum can be written as

$$E = \frac{1}{2}\Big(s^2 + s(2|\tilde{g}| + 1) + |\tilde{g}|\Big). \tag{2.37}$$

The lowest energy corresponds to $s = 0$. When $\tilde{g} \geq 0$, ground states are described by anti-holomorphic functions of degree $2\tilde{g}$ because $s = 0$ in that case implies $m = 0$, $n = 2\tilde{g}$. When $\tilde{g} \leq 0$, we find that ground states are described by holomorphic functions of degree $2|\tilde{g}|$. The number of independent ground states can be evaluated by counting the number of independent components of totally symmetric coefficient tensors of degree $2|\tilde{g}|$. Since each index can take two values there are $2|\tilde{g}|+1$ independent ground states. Higher energy states can be similarly obtained.

It is also interesting to look for the relations between the Hamiltonian and angular momentum squared. Using Eqs. (2.28) and (2.29), we find that eq. (2.21) becomes

$$K_aK_a = 2H + \tilde{g}^2. \tag{2.38}$$

The eigenvalues of the angular momentum squared can be evaluated from this equation using the energy spectrum (2.37) and, as expected, the result turns out to be $k(k+1)$ with

$$k = k_{\min} + s\,, \qquad k_{\min} = |\tilde{g}|. \tag{2.39}$$

Note that if $|\tilde{g}|$ is a half integer (or an integer), so must be k. In terms of this angular momentum quantum number the energy spectrum can be written as

$$E = \frac{1}{2}\left(k(k+1) - \tilde{g}^2\right). \tag{2.40}$$

The ground state energy $E_{\rm min} = \frac{1}{2}|\tilde{g}|$ is achieved when the angular momentum quantum number takes the minimum value $k = |\tilde{g}|$. Since there should be $2k+1$ degenerate states for a given k, it follows that the ground state degeneracy is due to the angular momentum degeneracy. The relation (2.40) agrees with the well known result obtained using the conventional approach [7] except that the usual monopole charge g is replaced with $\tilde{g}$, which could be interpreted as the effective monopole charge. The effect of the ordering parameter α_G can be absorbed if we redefine the monopole charge. In particular, $\tilde{g}$ is the same as g if $\alpha_G = -\frac{3}{2}$ is chosen.

3. $N = 2$ supersymmetric particle

We next extend the previous analysis to $N = 2$ supersymmetric case. $N = 2$ supersymmetric monopole Lagrangian [12] is given by

$$L = 2|\dot{z} - z(\bar{z}\cdot\dot{z})|^2 + \frac{i}{2}(\bar{\psi}\cdot\dot{\psi} - \dot{\bar{\psi}}\cdot\psi) - \frac{i}{2}(\bar{z}\cdot\dot{z} - \dot{\bar{z}}\cdot z)\bar{\psi}\cdot\psi + ig(\bar{z}\cdot\dot{z} - \dot{\bar{z}}\cdot z - i\bar{\psi}\cdot\psi), \quad (3.1)$$

where in addition to the bosonic degrees of freedom z_i there are also anti-commuting fermionic degrees of freedom denoted by ψ_i. As before, the dots between the symbols mean contractions of the indices.

The momenta p and $\bar{p}$ conjugate to z and $\bar{z}$, respectively, are given by

$$p = 2\left(\dot{\bar{z}} - (\dot{\bar{z}}\cdot z)\bar{z}\right) - \frac{i}{2}(\bar{\psi}\cdot\psi - 2g)\bar{z}\,, \qquad \bar{p} = 2\left(\dot{z} - z(\bar{z}\cdot\dot{z})\right) + \frac{i}{2}(\bar{\psi}\cdot\psi - 2g)z, \quad (3.2)$$

and the Hamiltonian is given by

$$H_c = 2|\dot{z} - z(\bar{z}\cdot\dot{z})|^2 - g\bar{\psi}\cdot\psi = \frac{1}{2}p_i A_{ij}\bar{p}_j - g\bar{\psi}\cdot\psi. \quad (3.3)$$

Due to supersymmetries, the bosonic constraints C_1 and C_2 of the previous section should be supplemented by two more fermionic constraints. They are obtained [12] by applying supertransformations on C_1. Altogether, there are four second class constraints

$$C_1 = \bar{z}\cdot z - 1\,, \qquad C_2 = p\cdot z + \bar{z}\cdot\bar{p}\,, \qquad C_3 = \bar{z}\cdot\psi\,, \qquad C_4 = \bar{\psi}\cdot z, \quad (3.4)$$

and one first class constraint corresponding to the Gauss law constraint

$$C_0 = -i(\bar{z}\cdot\bar{p} - p\cdot z) - \bar{\psi}\cdot\psi + 2g. \quad (3.5)$$

The Poisson brackets are defined as usual

$$\{z_i, p_j\} = \{\bar{z}_i, \bar{p}_j\} = \delta_{ij}\,, \qquad \{\bar{\psi}_i, \psi_j\} = -i\delta_{ij}, \quad (3.6)$$

and the Dirac brackets can be easily computed using the formula (2.10). The commutation relations consistent with the resulting Dirac brackets can be written as

$$\begin{aligned}
&[p_i, z_j] = -i\delta_{ij} + \frac{i}{2}\bar{z}_i z_j, \qquad && [p_i, \bar{z}_j] = \frac{i}{2}\bar{z}_i\bar{z}_j, \\
&[p_i, p_j] = \frac{i}{2}(p_i\bar{z}_j - p_j\bar{z}_i), \qquad && [\bar{p}_i, p_j] = \frac{i}{2}(\bar{z}_j\bar{p}_i - z_i p_j) + \bar{\psi}_j\psi_i - \alpha_F A_{ij}, \\
&\left[\bar{\psi}_i, \psi_j\right] = A_{ji}, && \left[p_i, \bar{\psi}_j\right] = i\bar{\psi}_i\bar{z}_j.
\end{aligned} \quad (3.7)$$

The square bracket between two fermionic operators should be interpreted as the anticommutator. Apart from the appearance of the fermion operators, the basic structure remains the same as in the bosonic case. However, there is a small difference worth mentioning. In contrast to the bosonic case, the commutator $[\bar{p}_i, p_j]$ acquires a term quadratic in the fermion operators, which causes a new ordering ambiguity. In fact, the requirement that the second class constraints commute with all other operators does not fix the ordering completely. We introduced a real parameter α_F in the commutator between p and $\bar{p}$ to reflect this new kind of ordering ambiguity. As in the bosonic case, the Gauss law constraint suffers from the ordering ambiguity and we write the quantum Gauss law constraint as

$$\hat{C}_0 = -i(\bar{z}\cdot\bar{p} - p\cdot z) - \bar{\psi}\cdot\psi + \alpha_G + 2g = 0. \tag{3.8}$$

It turns out that the commutation relations (3.7) become greatly simplified if we introduce the following variables:

$$\begin{aligned} &\beta = \epsilon_{ij} z_j \psi_i, \qquad \bar{\beta} = \epsilon_{ij}\bar{\psi}_i \bar{z}_j, \\ &w_i = p_i - \tfrac{i}{2}\bar{z}_i\bar{\psi}\psi, \qquad \bar{w}_i = \bar{p}_i + \frac{i}{2} z_i \bar{\psi}\psi. \end{aligned} \tag{3.9}$$

This amounts to solving the fermionic constraints because the old variables automatically satisfying the constraints C_3 and C_4 can be readily recovered by the formula

$$\psi_i = \epsilon_{ij}\bar{z}_j\beta\,, \qquad \bar{\psi}_i = \epsilon_{ij}\bar{\beta} z_j. \tag{3.10}$$

In terms of these variables the commutation relations (3.7) can be written as

$$\begin{aligned} &[w_i, z_j] = -i\delta_{ij} + \frac{i}{2}\bar{z}_i z_j, \qquad [w_i, \bar{z}_j] = \frac{i}{2}\bar{z}_i\bar{z}_j, \\ &[w_i, w_j] = -\frac{i}{2}(\bar{z}_i w_j - \bar{z}_j w_i), \quad [\bar{w}_i, w_j] = \frac{i}{2}(\bar{z}_j\bar{w}_i - z_i w_j) - \alpha_F A_{ij}, \\ &[w_i, \beta] = 0, \qquad\qquad [w_i, \bar{\beta}] = 0, \\ &[\beta, \bar{\beta}] = 1, \end{aligned} \tag{3.11}$$

and the constraints as

$$C_1 = \bar{z}\cdot z - 1\,, \qquad C_2 = \bar{z}\cdot\bar{w} + w\cdot z\,, \qquad \hat{C}_0 = -i(\bar{z}\cdot\bar{w} - w\cdot z) - 2\bar{\beta}\cdot\beta + \alpha_G + 2g. \tag{3.12}$$

Note that the bosonic and fermionic sectors completely decouple from each other, and β and $\bar{\beta}$ play the role of annihilation and creation operators in the fermionic sector. Note also that the operators $(w, \bar{w})$ satisfy the same commutation relations as $(p, \bar{p})$ in eq. (2.12) except the term involving α_F. Due to this similarity, we can almost repeat the analysis of the bosonic case. Namely, we decompose w_i into two parts,

$$U_B = -i(\bar{z}\cdot\bar{w} - w\cdot z)\,, \qquad B_i = w_i + \frac{i}{2} U_B \bar{z}_i. \tag{3.13}$$

The second class constraints are again given by eq. (2.15), and the Gauss law constraint $\hat{C}_0$ in eq. (3.8) becomes

$$U_B - 2\bar{\beta}\beta + \alpha_G + 2g = 0. \tag{3.14}$$

Most of the commutation relations (2.17) remain the same. The only difference is that the last equation is modified to

$$\left[B_i, \bar{B}_j\right] = -\left(U_B - \alpha_F - \frac{1}{2}\right) A_{ji}. \tag{3.15}$$

The supercharges are given by

$$Q = p \cdot \psi = \epsilon_{ij} B_i \bar{z}_j \beta = a\beta\,, \qquad \bar{Q} = \bar{\psi} \cdot \bar{p} = \bar{\beta}\epsilon_{ij} z_j \bar{B}_i = \bar{\beta}\bar{a}, \tag{3.16}$$

where a and $\bar{a}$ defined as in eq. (2.31) satisfy the same form of commutation relations

$$[a, \bar{a}] = -\tilde{U}_B\,, \qquad \left[\tilde{U}_B, a\right] = -2a\,, \qquad \left[\tilde{U}_B, \bar{a}\right] = 2\bar{a}, \tag{3.17}$$

if we define $\tilde{U}_B$ by

$$\tilde{U}_B \equiv U_B - \alpha_F - \frac{3}{2}. \tag{3.18}$$

Note the difference from eq. (2.29). Here, we have absorbed the ordering parameter α_F into the definition of the bosonic $U(1)$ generator. We choose as our quantum Hamiltonian

$$H = \frac{1}{2}\left[Q, \bar{Q}\right] = \frac{1}{2}\left(a\bar{a} - [a, \bar{a}]\,\bar{\beta} \cdot \beta\right). \tag{3.19}$$

It can be shown that this supersymmetric Hamiltonian agrees with the classical expression (3.3) up to an ordering term.

We now proceed to construct the Hilbert space representation. The Hilbert space consists of column vectors of the form

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \tag{3.20}$$

where each entry belongs to the bosonic Hilbert space considered in the previous section. The Hermitian product is trivially extended. On this Hilbert space the fermionic operators β and $\bar{\beta}$ are represented by the matrices

$$\beta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \bar{\beta} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{3.21}$$

Bosonic operators B_i and $\bar{B}_i$ are represented, as before, by eq. (2.25), and their commutator compared with eq. (3.15) yields the following identification:

$$\tilde{U}_B = z_k \frac{\partial}{\partial z_k} - \bar{z}_k \frac{\partial}{\partial \bar{z}_k}. \tag{3.22}$$

The $N = 2$ supersymmetric Hamiltonian is related to our bosonic Hamiltonian (2.32) as follows:

$$H = \frac{1}{4}\left(a\bar{a} + \bar{a}a\right) - \frac{1}{2}\left[a, \bar{a}\right]\Sigma = \frac{1}{4}\left(a\bar{a} + \bar{a}a\right) - \tilde{g}\Sigma + \frac{1}{4}, \tag{3.23}$$

where we define the spin operator

$$\Sigma \equiv \bar{\beta}\beta - \frac{1}{2}, \tag{3.24}$$

which has $\sigma = \pm(1/2)$ as its eigenvalues. In matrix form, the Hamiltonian can be written as

$$H = \frac{1}{2}\begin{pmatrix} \bar{a}a & 0 \\ 0 & a\bar{a} \end{pmatrix}. \tag{3.25}$$

We write the Gauss law constraint (3.14) as[3]

$$\tilde{U}_B - 2\Sigma + 2\tilde{g} = 0, \tag{3.26}$$

with

$$2\tilde{g} \equiv \frac{1}{2} + \alpha_F + \alpha_G + 2g. \tag{3.27}$$

Note that the definition of the effective magnetic charge $\tilde{g}$ is different[4] from Eq. (2.29) in bosonic case. In terms of the eigenvalues Eq. (3.26) can be rewritten as

$$(m - n) + 2(\tilde{g} - \sigma) = 0. \tag{3.28}$$

This implies that $2\tilde{g}$ should be an integer, and if $2g$ is also an integer that the sum of the two ordering parameters $\alpha_F + \alpha_G$ should be a half-integer.

To obtain the energy spectrum apply the Hamiltonian (3.23) to the components of the column vector (3.20). From Eqs. (2.32), (2.36) and (3.22), we find

$$E = \frac{1}{4}(2mn + m + n) + \frac{1}{2}(m - n)\sigma, \tag{3.29}$$

where $\sigma = 1/2$ for the upper component and $\sigma = -1/2$ for the lower component. As in the previous section we set $m = s$, $n = s + 2(\tilde{g} - \sigma)$ if $\tilde{g} - \sigma \geq 0$, and $n = s$, $m = s - 2(\tilde{g} - \sigma)$ if $\tilde{g} - \sigma \leq 0$, where $s = 0, 1, 2, \cdots$. Then the energy spectrum can be written as

$$E = \frac{1}{2}\Big(s^2 + s(2|\tilde{g} - \sigma| + 1) + |\tilde{g} - \sigma|\Big) - (\tilde{g} - \sigma)\sigma. \tag{3.30}$$

This energy spectrum can be written in simple form if we use the angular momentum quantum number. For this purpose, define the angular momentum operator K_a by

$$K_a = \frac{i}{2}\left(\bar{z}\sigma_a\bar{B} - B\sigma_a z + 2i\,\bar{z}\sigma_a z\right) - \frac{1}{2}\left(U_B - \alpha_F - \frac{3}{2}\right)\bar{z}\sigma_a z, \tag{3.31}$$

which satisfies the commutation relations (2.20). Note that the parameter α_F appears because of its presence in the commutation relations (3.11). Calculation similar to eq. (2.21)

[3]It can be shown that $\tilde{U}_F \equiv -2\Sigma$ is the fermionic $U(1)$ generator and $\tilde{U}_B + \tilde{U}_F$ is the total $U(1)$ generator. Note that the definition of $\tilde{g}$ here differs from that of ref. [12] by a shift of 1/2.

[4]To avoid confusion with notations it is important to remember that we are using the same symbol if their physical meaning is the same but their definitions may be different depending on what kind of particle we are considering. In general, we will not repeat writing the definition if it is the same as the previous one.

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yields

$$K^2 = 2H + \frac{1}{4}\left(\tilde{U}_B - 2\Sigma - 1\right)\left(\tilde{U}_B - 2\Sigma + 1\right)$$
$$= 2H + \left(\tilde{g} - \frac{1}{2}\right)\left(\tilde{g} + \frac{1}{2}\right). \tag{3.32}$$

From this equation and the energy spectrum (3.30) we find that the eigenvalue of the squared angular momentum is $k(k+1)$ with

$$k = k_{\min} + s\,, \qquad k_{\min} = |\tilde{g} - \sigma|. \tag{3.33}$$

Conversely, the energy spectrum can be written in terms of k as

$$E = \frac{1}{2}\left(k(k+1) - \left(\tilde{g} - \frac{1}{2}\right)\left(\tilde{g} + \frac{1}{2}\right)\right). \tag{3.34}$$

Using this result we easily find that zero energy is achieved by the upper component if $\tilde{g} \geq \frac{1}{2}$ and by the lower component if $\tilde{g} \leq -\frac{1}{2}$. We list below some of the few zero energy states:

$$\begin{array}{ccccccccc} \cdots & \begin{pmatrix} 0 \\ zz \end{pmatrix} & \begin{pmatrix} 0 \\ z \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} \bar{z} \\ 0 \end{pmatrix} & \begin{pmatrix} \bar{z}\bar{z} \\ 0 \end{pmatrix} & \cdots, \\ & \tilde{g} = -\frac{3}{2} & \tilde{g} = -1 & \tilde{g} = -\frac{1}{2} & \tilde{g} = \frac{1}{2} & \tilde{g} = 1 & \tilde{g} = \frac{3}{2} & \end{array} \tag{3.35}$$

where we omitted all the indices, indicating only the polynomial nature of the state on z and $\bar{z}$. The number of degeneracies for these states is $2|\tilde{g} - \sigma| + 1 = 2k + 1$. Excited states can be similarly constructed.

The case with $\tilde{g} = 0$ is somewhat special because the energy spectrum is $E = \frac{1}{2}(s+1)^2$ for both upper and lower components. This means that there is no state invariant under the supersymmetry transformations. The minimum energy sector in this case consists of two copies of $k = \frac{1}{2}$ states

$$\begin{pmatrix} 0 \\ \bar{z} \end{pmatrix}, \qquad \begin{pmatrix} z \\ 0 \end{pmatrix}, \tag{3.36}$$

which are related to each other by the supersymmetries. The supersymmetry is spontaneously broken for $\tilde{g} = 0$.

4. $N = 4$ supersymmetric particle

The Lagrangian for $N = 4$ superparticle moving on S^2 is given [13] by

$$L = 2|\dot{z} - z(\bar{z}\cdot\dot{z})|^2 + \frac{i}{2}(\bar{\psi}_\alpha\cdot\dot{\psi}_\alpha - \dot{\bar{\psi}}_\alpha\cdot\psi_\alpha) - \frac{i}{2}(\bar{z}\cdot\dot{z} - \dot{\bar{z}}\cdot z)\bar{\psi}_\alpha\cdot\psi_\alpha$$
$$- \frac{1}{2}(\bar{\psi}_\alpha\cdot\psi_\alpha)^2 + ig(\bar{z}\cdot\dot{z} - \dot{\bar{z}}\cdot z - i\bar{\psi}_\alpha\cdot\psi_\alpha), \tag{4.1}$$

where the fermion field now carries an additional index $\alpha = (1, 2)$. Note that this Lagrangian differs from $N = 2$ Lagrangian (3.1) by the presence of the quartic fermionic interaction term which is essential for the existence of $N = 4$ supersymmetry.

Canonical quantization of the system goes in parallel with that of $N = 2$ system. Additional fermion indices are treated in an obvious manner. Momenta p and $\bar{p}$ conjugate, respectively, to the fields z and $\bar{z}$ are

$$p = 2\left(\dot{\bar{z}} - (\dot{\bar{z}} \cdot z)\bar{z}\right) - \frac{i}{2}(\bar{\psi}_\alpha \cdot \psi_\alpha - 2g)\bar{z}\,, \qquad \bar{p} = 2\left(\dot{z} - z(\bar{z} \cdot \dot{z})\right) + \frac{i}{2}(\bar{\psi}_\alpha \cdot \psi_\alpha - 2g)z. \tag{4.2}$$

The classical Hamiltonian is

$$H_c = 2|\dot{z} - z(\bar{z} \cdot \dot{z})|^2 + \frac{1}{2}(\bar{\psi}_\alpha \cdot \psi_\alpha)^2 - g\bar{\psi}_\alpha \cdot \psi_\alpha. \tag{4.3}$$

Constraints are trivially extended. We have the following six second class constraints

$$C_1 = \bar{z} \cdot z - 1\,, \qquad C_2 = p \cdot z + \bar{z} \cdot \bar{p}\,, \qquad C_{3\alpha} = \bar{z} \cdot \psi_\alpha\,, \qquad C_{4\alpha} = \bar{\psi}_\alpha \cdot z, \tag{4.4}$$

and one first class constraint,

$$C_0 = -i(\bar{z} \cdot \bar{p} - p \cdot z) - \bar{\psi}_\alpha \cdot \psi_\alpha + 2g. \tag{4.5}$$

Starting with the Poisson bracket relations

$$\{z_i, p_j\} = \{\bar{z}_i, \bar{p}_j\} = \delta_{ij}\,, \qquad \{\bar{\psi}_{i\alpha}, \psi_{j\beta}\} = -i\delta_{ij}\delta_{\alpha\beta}, \tag{4.6}$$

we quantize the system according the Dirac scheme to find the following quantum commutation relations

$$\begin{aligned}
&[p_i, z_j] = -i\delta_{ij} + \frac{i}{2}\bar{z}_i z_j, \qquad [p_i, \bar{z}_j] = \frac{i}{2}\bar{z}_i\bar{z}_j, \\
&[p_i, p_j] = \frac{i}{2}(p_i\bar{z}_j - p_j\bar{z}_i), \qquad [\bar{p}_i, p_j] = \frac{i}{2}(\bar{z}_j\bar{p}_i - z_i p_j) + \bar{\psi}_{j\alpha}\psi_{i\alpha} - \alpha_F A_{ij}, \\
&\left[\bar{\psi}_{i\alpha}, \psi_{j\beta}\right] = \delta_{\alpha\beta}A_{ji}, \qquad \left[p_i, \bar{\psi}_{j\alpha}\right] = i\bar{\psi}_{i\alpha}\bar{z}_j.
\end{aligned} \tag{4.7}$$

They form a straightforward generalization of eq. (3.7).

We then solve the constraints $C_{3\alpha}$ and $C_{4\alpha}$, as before, by introducing β_α and $\bar{\beta}_\alpha$ as

$$\beta_\alpha = \epsilon_{ij} z_j \psi_{i\alpha}\,, \qquad \bar{\beta}_\alpha = \epsilon_{ij}\bar{z}_j\bar{\psi}_{i\alpha}, \tag{4.8}$$

and define w_i and $\bar{w}_i$ by,

$$w_i = p_i - \frac{i}{2}\bar{z}_i\bar{\psi}_\alpha\psi_\alpha\,, \qquad \bar{w}_i = \bar{p}_i + \frac{i}{2}z_i\bar{\psi}_\alpha\psi_\alpha. \tag{4.9}$$

Bosonic part of the commutation relations remains the same as eq. (3.11) and the commutators invoving fermions become

$$[w_i, \beta_\alpha] = 0\,, \qquad \left[w_i, \bar{\beta}_\alpha\right] = 0\,, \qquad \left[\bar{\beta}_\alpha, \beta_\beta\right] = \delta_{\alpha\beta}. \tag{4.10}$$

This shows that the fermion sector again decouples from the bosonic one and the number of fermion annihilation and creation operators is doubled.

We can proceed to introduce U_B and B_i and represent them on the Hilbert space as in $N = 2$ case. Because there are two fermion creation operators the number of components of state vectors is increased to four. Fermion operators $\bar{\beta}_\alpha$ and β_α can be represented by 4×4 matrices as follows:

$$\beta_1 = \begin{pmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&-1&0&0 \end{pmatrix}, \quad \bar{\beta}_1 = \begin{pmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix}, \quad \bar{\beta}_2 = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&-1 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}. \tag{4.11}$$

The supercharges are given by

$$Q_\alpha = p \cdot \psi_\alpha = \epsilon_{ij} B_i \bar{z}_j \beta_\alpha \equiv a\beta_\alpha \,, \qquad \bar{Q}_\alpha = \bar{\psi}_\alpha \cdot \bar{p} = \bar{\beta}_\alpha \epsilon_{ij} z_j \bar{B}_i \equiv \bar{\beta}_\alpha \bar{a}, \tag{4.12}$$

and satisfy the commutation relation

$$\left[Q_\alpha, \bar{Q}_\beta\right] = a\bar{a}\delta_{\alpha\beta} - [a, \bar{a}]\, \bar{\beta}_\beta \beta_\alpha. \tag{4.13}$$

Unlike $N = 2$ case, the $N = 4$ supersymmetric Hamiltonian cannot be obtained simply by taking the trace of this equation because the result does not commute with the supercharges. Nevertheless, it can be shown[5] that the Hamiltonian can be defined as

$$H = \frac{1}{4}\left[Q_\alpha, \bar{Q}_\alpha\right] - \frac{1}{2}\tilde{g}\Sigma, \tag{4.14}$$

where Σ and $\tilde{g}$ are defined by

$$\Sigma \equiv \bar{\beta}_\alpha \beta_\alpha - 1, \qquad 2\tilde{g} \equiv 2g + \alpha_G + \alpha_F - \frac{1}{2}, \tag{4.15}$$

which differ from the corresponding equations (3.24) and (3.27) in $N = 2$ case. Note that the $N = 4$ spin operator Σ has eigenvalues $\sigma = (1, 0, 0, -1)$. The Gauss law constraint maintains the same form as eqs. (3.26) and (3.28). From eq. (4.13) we get

$$\begin{aligned} \frac{1}{4}\left[Q_\alpha, \bar{Q}_\alpha\right] &= \frac{1}{2}a\bar{a} - \frac{1}{4}[a, \bar{a}]\, \bar{\beta}_\alpha \beta_\alpha \\ &= \frac{1}{4}(a\bar{a} + \bar{a}a) - \frac{1}{4}[a, \bar{a}]\, \Sigma. \end{aligned} \tag{4.16}$$

Inserting this equation into eq. (4.14) and using the Gauss law we can write Hamiltonian in the following form:

$$H = \frac{1}{4}(a\bar{a} + \bar{a}a) - \tilde{g}\Sigma + \frac{1}{2}\Sigma^2. \tag{4.17}$$

In matrix form, it becomes

$$H = \frac{1}{4}(a\bar{a} + \bar{a}a) + \begin{pmatrix} \frac{1}{2} - \tilde{g} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} + \tilde{g} \end{pmatrix}. \tag{4.18}$$

[5]This hamiltonian differs from the one used in ref. [13] by the constant term $\frac{1}{2}\tilde{g}$. See the discussion in section 5.

The energy spectrum immediately follows from eq. (4.17)

$$E = \frac{1}{4}(2mn + m + n) - \tilde{g}\sigma + \frac{1}{2}\sigma^2, \tag{4.19}$$

which in terms of the notation used in Eqs. (2.37) and (3.30) can be written as

$$E = \frac{1}{2}\Big(s^2 + s(2|\tilde{g} - \sigma| + 1) + |\tilde{g} - \sigma|\Big) - \tilde{g}\sigma + \frac{1}{2}\sigma^2. \tag{4.20}$$

It is useful to express the energy spectrum in terms of the angular momentum quantum number. It turns out that the angular momentum operator K_a in $N = 4$ case has the same expression as Eq. (3.31) and the calculation of its square yields

$$K^2 = 2H + \tilde{g}^2. \tag{4.21}$$

Using Eq. (4.20) we again find that the spectrum for K^2 is $k(k+1)$, where

$$k = k_{\min} + s\,, \qquad k_{\min} = |\tilde{g} - \sigma|. \tag{4.22}$$

Conversely, the energy spectrum can be written as

$$E = \frac{1}{2}\left(k(k+1) - \tilde{g}^2\right). \tag{4.23}$$

For a given $\tilde{g}$, $E_{\min}$ is determined by $k_{\min}$. We tabulate $k_{\min}$, $E_{\min}$ and by which states these values are achieved for each values of $\tilde{g}$.

$$\begin{aligned}
&k_{\min} = |\tilde{g}| - 1, && E_{\min} = -\frac{1}{2}|\tilde{g}|, && \sigma = -1, \text{ for } \tilde{g} \le -1,\\
&k_{\min} = \frac{1}{2}, && E_{\min} = \frac{1}{2}, && \left\{\begin{matrix}\sigma = -1\\ \sigma = 0\end{matrix}\right\}, \text{ for } \tilde{g} = -\frac{1}{2},\\
&k_{\min} = 0, && E_{\min} = 0, && \sigma = 0, \quad \text{for } \tilde{g} = 0,\\
&k_{\min} = \frac{1}{2}, && E_{\min} = \frac{1}{2}, && \left\{\begin{matrix}\sigma = 0\\ \sigma = +1\end{matrix}\right\}, \text{ for } \tilde{g} = +\frac{1}{2},\\
&k_{\min} = |\tilde{g}| - 1, && E_{\min} = -\frac{1}{2}|\tilde{g}|, && \sigma = +1, \text{ for } \tilde{g} \ge +1.
\end{aligned} \tag{4.24}$$

We list below supersymmetric ground states for a few values of $\tilde{g}$:

$$\begin{array}{ccccccccc}
\cdots & \begin{pmatrix}0\\0\\0\\zz\end{pmatrix} & \begin{pmatrix}0\\0\\0\\z\end{pmatrix} & \begin{pmatrix}0\\0\\0\\1\end{pmatrix} & \begin{pmatrix}0\\0\\1\\0\end{pmatrix} & \begin{pmatrix}0\\1\\0\\0\end{pmatrix} & \begin{pmatrix}1\\0\\0\\0\end{pmatrix} & \begin{pmatrix}\bar{z}\\0\\0\\0\end{pmatrix} & \begin{pmatrix}\bar{z}\bar{z}\\0\\0\\0\end{pmatrix} & \cdots \\
& \tilde{g} = -2 & \tilde{g} = -\frac{3}{2} & \tilde{g} = -1 & \tilde{g} = 0 & \tilde{g} = 0 & \tilde{g} = 1 & \tilde{g} = \frac{3}{2} & \tilde{g} = 2
\end{array} \tag{4.25}$$

For each values of $\tilde{g}$ there are $2k+1$ independent states, again showing that the degeneracy is entirely due to the angular momentum degeneracy. Consider $\tilde{g} = -2$ case for instance.

Since $s = 0$ and $\sigma = -1$ for these states, the angular momentum quantum number should be $k = |\tilde{g} - \sigma| = 1$, and $2k + 1 = 3$ agrees with the number of independent states given by symmetric combinations $z_1 z_1$, $z_2 z_2$ and $z_1 z_2 + z_2 z_1$.

For $\tilde{g} = \pm\frac{1}{2}$ there does not exist any state which is invariant under the full $N = 4$ supersymmetry because the minimum energy $E_{\min} = \frac{1}{2}$ is greater than $-\frac{1}{2}\tilde{g}\Sigma$, the energy value supersymmetric invariant states should have as can be seen from eq. (4.14). The ground states for these values are given by

$$\begin{pmatrix} 0 \\ 0 \\ z \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ z \\ 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \\ \bar{z} \end{pmatrix} \qquad \text{for } \tilde{g} = -\tfrac{1}{2}, \tag{4.26}$$

and

$$\begin{pmatrix} 0 \\ 0 \\ \bar{z} \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ \bar{z} \\ 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} z \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{for } \tilde{g} = \tfrac{1}{2}, \tag{4.27}$$

consisting of three copies of $k = \frac{1}{2}$ states, six of them altogether. They are related by supersymmetric transformations. For $\tilde{g} = -1/2$, for instance, the first and the third states are related by Q_1 and $\bar{Q}_1$ and the second and the third states are related by Q_2 and $\bar{Q}_2$ as in the following diagrams:

$$\begin{pmatrix} 0 \\ 0 \\ -iz_2 \\ 0 \end{pmatrix} \quad \begin{matrix} \bar{Q}_1 \\ \longleftarrow \\ \longrightarrow \\ Q_1 \end{matrix} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ \bar{z}_1 \end{pmatrix} \quad \begin{matrix} \bar{Q}_2 \\ \longrightarrow \\ \longleftarrow \\ Q_2 \end{matrix} \quad \begin{pmatrix} 0 \\ iz_2 \\ 0 \\ 0 \end{pmatrix}, \tag{4.28}$$

$$\begin{pmatrix} 0 \\ 0 \\ iz_1 \\ 0 \end{pmatrix} \quad \begin{matrix} \bar{Q}_1 \\ \longleftarrow \\ \longrightarrow \\ Q_1 \end{matrix} \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ \bar{z}_2 \end{pmatrix} \quad \begin{matrix} \bar{Q}_2 \\ \longrightarrow \\ \longleftarrow \\ Q_2 \end{matrix} \quad \begin{pmatrix} 0 \\ -iz_1 \\ 0 \\ 0 \end{pmatrix}. \tag{4.29}$$

Note that the states in the second column of the above diagram are not invariant under any real supertransformations. On the other hand, the states on the left are killed by Q_2 and $\bar{Q}_2$, and the ones on the right are annihilated by Q_1 and $\bar{Q}_1$. From this fact we conclude that the space of the ground states for $\tilde{g} = \pm 1/2$ consists of a two dimensional subspace consisting of the states not invariant under any supersymmetries and a four dimensional subspace consisting of the states invariant under $N = 2$ supersymmetry.

5. Summary and discussions

We have presented a complete solution to the quantum mechanical problem of a charged particle moving on S^2 in the background of a magnetic monopole at the center, starting

with the simplest case of a bosonic particle and extending the results to the supersymmetric cases. In studying this model we have used $CP(1)$ type of coordinates. This choice of coordinates has a certain advantage over the conventional one. On the other hand, the use of redundant coordinates produces more constraints which make the quantization difficult. In principle, transition from the classical Dirac brackets to the quantum commutation relations is not unique due to the operator ordering ambiguity. Moreover, quantization of the constraints can also involve ordering ambiguities. In this work we have carefully retained all the possible ordering terms and found certain quantization conditions they have to satisfy and eventually shown that their effects can be absorbed by redefining the magnetic charge. The quantum Hamiltonian may also have operator ordering ambiguities. In our model, after using the quantum Gauss law constraint the ordering ambiguity amounts to adding a constant term linear in the magnetic charge. We have chosen the Hamiltonian in such a way that the energy spectrum respects the symmetry under the simultaneous flip of the magnetic field and the spin, which certainly holds in the classical model. We have also required the minimum energy to be zero when $\tilde{g}$ vanishes. This condition further fixes $\tilde{g}$-independent constant term.

We have constructed the Hilbert space representation of the fundamental quantum commutation relations, which was lacking in the previous work of refs. [12, 13]. Using this representation we have found the complete energy eigenfunctions. In particular, the ground states were studied in detail. Explicit functional forms were presented and the number of degeneracies were counted. For those values of $\tilde{g}$ for which the ground states are invariant under all supersymmetries we have shown that the number of degeneracy is $2k_{\min}+1$. In refs. [12, 13] it was noted that for certain values of $\tilde{g}$, i.e., $\tilde{g}=0$ in $N=2$ case and $\tilde{g}=\pm\frac{1}{2}$ in $N=4$ case, the supersymmetry is spontaneously broken. In this work we have further investigated the ground state structure for these particular values of $\tilde{g}$ and it was shown that the ground states consists of two copies of $k=\frac{1}{2}$ states in $N=2$ case and three copies of $k=\frac{1}{2}$ states in $N=4$ case. It was further shown, in $N=4$ case, that one of them is not invariant under any supersymmetry transformation and the remaining two are invariant under a half of the supersymmetry transformations. Also in $N=4$ case, we notice from eq. (4.24) that the energy of the supersymmetric ground states for $|\tilde{g}|\geq 1$ is negative, $E_{\min}=-\frac{1}{2}|\tilde{g}|$, which was not possible in the $N=2$ case. This is a special feature of the $N=4$ system due to the last term of eq. (4.14). A similar type of relation appears also in ref. [18]. In our model, it can be roughly explained by saying that the magnetic field and spin interaction term in eq. (4.17) becomes dominant over the spin-spin interaction term for large values of $|\tilde{g}|$.

There are several aspects of this work which deserve further studies. It seems possible to extend our analysis to any number of supersymmetries beyond $N=4$ and it would be interesting to see how the symmetry breaking pattern continues. Next, noting that in our quantum mechanics model supersymmetry is spontaneously broken for some special values of the effective monopole charge $\tilde{g}$, it would be interesting to investigate the issue of spontaneous supersymmetry breaking in the field theoretical extensions of our model, paying attentions to the role of operator ordering ambiguity and checking whether these particular values of $\tilde{g}$ have any special meaning. Another topic is to consider the system on the fuzzy

(super) sphere [19] and analyze whether some new features of spontaneous supersymmetry breaking occur on the fuzzy (super) sphere. It would be also worth investigating the BRST extension [20] of our supersymmetric monopole system in which the ordering ambiguities could be further addressed.

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